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Bounds on the trace mapping of *LD*-fields

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Abstract

Bounds on the trace mappings defined on the Sobolev space $W_1^1(\Omega)$ and the space $LD(\Omega)$ of integrable strains are obtained. In the mechanics of continuous media, such bounds correspond to stress concentration—the ratio between the maximal stress in a body and the maximum of the traction applied to its boundary. The analysis leading to the bounds may be described in the mechanical context of stress theory and stress concentration.

Keywords: Trace; Bounds; Continuum mechanics; Stress concentration; Sobolev space

1. Introduction

This work considers some mathematical aspects of stress concentration. Stress concentration is a term used by engineers to indicate the increase of stresses from some expected values due to deviation of the geometry of the body from an idealized simple one. In practice, stress concentration factors are used by engineers to indicate the ratio between the maximum of the stress for the actual body under consideration and the maximal stresses calculated for the simplified geometry for which formulae of strength-of-materials are used traditionally. Stress concentration factors are usually compiled for homogeneous, linear, isotropic, elastic solids for various typical geometries (see for example [1]). Their values are obtained by solving the equations of elasticity analytically, by numerical approximations and by experimental methods.

In recent work, [2,3], we formalized the notion of stress concentration mathematically and generalized it. The idea in formalizing the stress concentration factor is to regard the stresses for the simplified geometry as traction boundary conditions. For the example of a bar under tension F, the "nominal" stresses F/A, where A is the cross section area, are regarded as boundary conditions at the ends of the bar in agreement with the engineering notion. Thus, for a given surface traction distribution t on the boundary of the body Ω and a stress tensor σ in equilibrium with t, the stress concentration factor is defined by

$$K_{t,\sigma} = \frac{\operatorname*{ess\,sup}_x |\sigma(x)|}{\operatorname*{ess\,sup}_y |t(y)|}, \quad x \in \Omega,\ y \in \partial\Omega. \tag{1.1}$$

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Here, $|\sigma(x)|$ is a the norm of the value of the stress at x. To evaluate it we use some norm on the vector space of matrices. The value of $K_{t,\sigma}$ clearly depends on the norm used and various norms are discussed in Section 4. Using the essential supremum, we ignore high stresses on sets of zero volume.

Noting that without specifying particular constitutive relations there is a class Σ_t of stress fields that are in equilibrium with t, we define the optimal stress concentration factor as

$$K_t = \inf_{\sigma \in \Sigma_t} K_{t,\sigma}. \tag{1.2}$$

Next, realizing that an engineer usually does not know a-priori the nature of the loads acting on a body exactly, the *generalized stress concentration factor* is defined as

$$K = \sup_t K_t = \sup_t \left\{ \inf_{\sigma \in \Sigma_t} \frac{\operatorname*{ess\,sup}_{x \in \Omega} |\sigma(x)|}{\operatorname*{ess\,sup}_{y \in \partial\Omega} |t(y)|} \right\} \tag{1.3}$$

where the supremum is taken over all traction fields – essentially bounded vector fields on $\partial\Omega$ – i.e., over all $t \in L^\infty(\partial\Omega, \mathbb{R}^3)$. It is noted that K is a purely geometric property of the body Ω.

In [2,3] we related the generalized stress concentration factor to the norms of the trace mappings on Sobolev spaces and LD-spaces. It turns out that for the formulation of equilibrium and stress theory, particularly in the context of stress concentration, the Sobolev space $W_1^1(\Omega, \mathbb{R}^3)$ and the related $LD(\Omega)$ space are especially useful. We recall that the space $LD(\Omega)$ contains integrable vector fields w such that the components of their associated stretching, or (infinitesimal, linear) strain,

$$\varepsilon(w) = \frac{1}{2}(\nabla w + (\nabla w)^{\mathrm{T}}), \qquad \varepsilon(w)_{ij} = \frac{1}{2}(w_{i,j} + w_{j,i}), \tag{1.4}$$

are also integrable (see [4–7]).

Assuming that Ω is an open subset of $\mathbb{R}^n$ having a C^2-boundary, for both spaces one has a well defined, linear, bounded trace mapping γ, such that for every vector field w defined on Ω, $\gamma(w)$ is a vector field defined on $\partial\Omega$ satisfying the following compatibility condition. For every continuous vector field u defined on the closure $\overline{\Omega}$, γ acts as the restriction to the boundary, i.e.,

$$\gamma(u|_\Omega) = u|_{\partial\Omega}. \tag{1.5}$$

In [2] we have shown that if we ignore the requirement that the total force and total torque on every subbody of Ω vanish, then,

$$K = \|\gamma\|, \quad \text{for } \gamma\colon W_1^1(\Omega, \mathbb{R}^3) \longrightarrow L^1(\partial\Omega, \mathbb{R}^3). \tag{1.6}$$

For the case where the total forces and torques on the various subbodies do vanish, a more detailed analysis is required (see [3]). Letting $\mathscr{R}$ be the finite dimensional vector space of rigid vector fields, one has to consider the quotient spaces $LD(\Omega)/\mathscr{R}$ and $L^1(\partial\Omega, \mathbb{R}^3)/\mathscr{R}$. For these spaces, one can define an induced trace mapping

$$\gamma/\mathscr{R}\colon LD(\Omega)/\mathscr{R} \longrightarrow L^1(\partial\Omega, \mathbb{R}^3)/\mathscr{R}, \tag{1.7}$$

and it turns out that

$$K = \|\gamma/\mathscr{R}\|. \tag{1.8}$$

Thus, apart from the mathematical interest in estimates on $\|\gamma\|$ and $\|\gamma/\mathscr{R}\|$, such estimates are very significant in stress analysis. It is our objective here to estimate these constants. Our method of estimation has a mechanical flavor. In a way, it is dual to the analysis leading to the relation between generalized stress concentration factors and the norms of the trace mappings.

For the particular case of the Sobolev space $W_1^1(\Omega)$, Motron obtained recently [8] some estimates on the bounds. The method used here is different and is based on the maximum principle for the Dirichlet problem. The bounds we obtain give a concrete estimate for $\|\gamma\|$. Subsequently, we extend our method to the space of LD-fields. Specifically, recalling that the LD-norm is given by

$$\|w\|_{LD} = \sum_i \|w_i\|_{L^1} + \sum_{i,m} \|\varepsilon(w)_{im}\|_{L^1}, \tag{1.9}$$

we obtain bounds on the constants A and B such that

$$\int_{\partial\Omega} |\gamma(w)| \leqslant A \int_{\Omega} |w| + B \int_{\Omega} |\varepsilon(w)| \tag{1.10}$$

for all LD-fields w (so that $\max\{A, B\}$ bounds $\|\gamma\|$).

Section 2 presents the results and methods of [3] relating stress concentration to the norm of the trace mapping. The last subsection discusses the simplification to the case of sourceless vector fields (rather then stress tensors) which may serve as motivation for studying the norm of the trace mapping for the Sobolev space $W_1^1(\Omega)$. In Section 3 we introduce the basic method for obtaining the bounds on the trace mapping for the Sobolev space using harmonic vector fields. The main result of this section is Theorem 3.4. As background material for the discussion of the bounds for the trace mapping on $LD(\Omega)$, Section 4 presents some standard results on norms of matrices. These are significant in the mechanical context as they are used on the space of stress matrices. For example, a yield criterion is usually a seminorm on the space of stress matrices. Section 5 presents additional preparatory material—the optimal boundary values for the stresses for a given boundary traction field. Section 6 studies the bounds on the norm of the trace mapping for $LD(\Omega)$ using the method of harmonic tensor fields and the central result is Theorem 6.1. Finally, the concluding remarks of Section 7 discuss the mechanical interpretation of the preceding analysis.

2. Generalized stress concentration and the norm of the trace mapping

2.1. Basic definitions and notation

We consider an open set $\Omega \subset \mathbb{R}^3$, where in some sections, the presentation is in the setting of $\mathbb{R}^n$. We assume that Ω is bounded and that it has a C^2-boundary. (The results hold for less restrictive assumptions.) We will use the summation convention for repeated indices and subscripted comma followed by an index will indicate partial differentiation with respect to the corresponding variable.

A vector field on Ω is interpreted physically as a virtual velocity field on the body or alternatively as a field of virtual infinitesimal displacements. A *rigid* field in $\mathbb{R}^3$ is a vector field of the form

$$w(x) = a + b \times x, \quad a, b \in \mathbb{R}^3.$$

Clearly, rigid fields may be restricted to subsets of $\mathbb{R}^3$. We denote the space of rigid fields by $\mathscr{R}$ and it is a 6-dimensional vector space.

The following definitions and results concerning LD-fields are due to Temam and Strang [4–6]. Given an integrable vector field w on Ω, we consider the corresponding *stretching* (*linear strain*) field $\varepsilon(w)$ defined by

$$\varepsilon(w)_{im} = \frac{1}{2}(w_{i,m} + w_{m,i}), \tag{2.1}$$

where the comma implies distributional derivative relative to the corresponding spatial coordinate. The integrable vector field w is of *integrable stretching*, or $w \in LD(\Omega)$, if the components of the corresponding stretching are also integrable over Ω. On the vector space $LD(\Omega)$ of integrable stretchings it is natural to use the norm

$$\|w\| = \|w\|_{LD} = \|w\|_1 + \|\varepsilon(w)\|_1, \tag{2.2}$$

where $\|\cdot\|_p$ indicates the L^p-norm. With this norm, $LD(\Omega)$ is a Banach space and we have a continuous and linear

$$\varepsilon\colon LD(\Omega) \longrightarrow L^1(\Omega, \mathbb{R}^6). \tag{2.3}$$

A basic theorem whose classical version is due to Liouville (see [6, pp. 18–19]) states:

Proposition 2.1. $\mathrm{Kernel}(\varepsilon) = \mathscr{R}$.

Let $\mathbf{W}$ be a Banach space of velocity fields. In the discussion below $\mathbf{W}$ will be either $LD(\Omega)$ or $L^1(\partial\Omega, \mathbb{R}^3)$. We refer to an element $\chi \in \mathbf{W}/\mathscr{R}$ as a *distortion*. We have the natural projection mapping onto the quotient space

$$\pi: \mathbf{W} \longrightarrow \mathbf{W}/\mathscr{R} \tag{2.4}$$

and the induced norm in $\mathbf{W}/\mathscr{R}$ is given by

$$\|\chi\| = \inf_{w\in\chi} \|w\|, \quad \text{or} \quad \|[w]\| = \inf_{r\in\mathscr{R}} \|w + r\|. \tag{2.5}$$

Proposition 2.2. *For both $\mathbf{W} = L^1(\partial\Omega, \mathbb{R}^3)$ and $\mathbf{W} = LD(\Omega)$ there are continuous and linear projection mappings*

$$\pi_{\mathscr{R}}: \mathbf{W} \longrightarrow \mathscr{R}. \tag{2.6}$$

For $w \in \mathbf{W}$, $\pi_{\mathscr{R}}(w) = a + b \times x$ is given by

$$a = \frac{1}{|U|}\int_U w, \qquad b = I^{-1}\left(\int_U x \times w\right), \tag{2.7}$$

where $U = \partial\Omega$ for $\mathbf{W} = L^1(\partial\Omega, \mathbb{R}^3)$, $U = \Omega$ for $\mathbf{W} = LD(\Omega)$, $|U|$ is the Hausdorff measure of U, and $I_{im} = \int_U (x_k x_k \delta_{im} - x_i x_m)$ is the moment of inertia of U.

The space $LD(\Omega)$ has the following properties (see [6]).
Approximation: The restrictions of fields in $C^\infty(\overline{\Omega}, \mathbb{R}^3)$ to Ω are dense in $LD(\Omega)$.
Extensions: There is a continuous linear extension operator $E: LD(\Omega) \to LD(\mathbb{R}^3)$.
Regularity: If w is any distribution on Ω whose corresponding stretching is L^1, then $w \in L^1(\Omega, \mathbb{R}^3)$.
Trace mapping: There is a unique linear, surjective, continuous trace mapping

$$\gamma: LD(\Omega) \longrightarrow L^1(\partial\Omega, \mathbb{R}^3) \tag{2.8}$$

such that $\gamma(w|_\Omega) = w|_{\partial\Omega}$ for all continuous vector fields w defined on Ω.
Distortions: On $LD(\Omega)/\mathscr{R}$,

$$\|\chi\|_\varepsilon = \|\varepsilon(\chi)\|_1 \tag{2.9}$$

is a norm that is equivalent to the quotient norm described above. Thus, there is a constant $C(\Omega)$ (depending on Ω only) such that for every $w \in LD(\Omega)$

$$\inf_{r\in\mathscr{R}} \|w + r\| \leqslant C(\Omega)\|\varepsilon(w)\|_1. \tag{2.10}$$

The infimum is attainable, i.e., for each $w \in LD(\Omega)$, there is a rigid motion r_0 satisfying

$$\|w + r_0\| = \inf_{r\in\mathscr{R}} \|w + r\| \leqslant C(\Omega)\|\varepsilon(w)\|_1. \tag{2.11}$$

Equivalent norm: If p is a seminorm on $LD(\Omega)$ such that $p(r) = 0$ implies that $r = 0$ for every $r \in \mathscr{R}$ (so p is a norm on $\mathscr{R}$), then,

$$p(w) + \|\varepsilon(w)\|_1 \tag{2.12}$$

is a norm on $LD(\Omega)$ that is equivalent to the original norm. In particular, using the trace mapping one can take

$$p(w) = \|\gamma(w)\|_{1,\partial\Omega}, \tag{2.13}$$

so the following is a norm that is equivalent to original (2.2)

$$\|w\|_\times = \|\gamma(w)\|_{1,\partial\Omega} + \|\varepsilon(w)\|_1. \tag{2.14}$$

Furthermore, one may use the projection $\pi_{\mathscr{R}}: L^1(\partial\Omega, \mathbb{R}^3) \to \mathscr{R}$ as in Proposition 2.2 and a norm $\|\cdot\|_{\mathscr{R}}$ to obtain the equivalent norm

$$\|w\|_\oplus = \|\pi_{\mathscr{R}} \circ \gamma(w)\|_{\mathscr{R}} + \|\varepsilon(w)\|_1. \tag{2.15}$$

Forces are regarded as elements of the dual spaces to the corresponding spaces of virtual velocities. So for a generic space of velocities $\mathbf{W}$, a force F will be a member of $\mathbf{W}^*$. The evaluation $F(w)$ is interpreted as virtual work, or virtual power, performed by the generalized force for the corresponding generalized velocity.

In the case the space $\mathbf{W}$ of velocities is an L^p-space, $1 \leqslant p \leqslant \infty$, (e.g., $L^1(\partial\Omega, \mathbb{R}^3)$) a force may be represented by an element of the corresponding L^q space with $q = p/(p-1)$ through integration. We will use the same symbol for the force and its representing field. For example, for $t \in L^1(\partial\Omega, \mathbb{R}^3)^* = L^\infty(\partial\Omega, \mathbb{R}^3)$ we have

$$t(w) = \int_{\partial\Omega} t \cdot w. \tag{2.16}$$

A force $F \in \mathbf{W}^*$ acting on a body is *equilibrated* if $F(r) = 0$ for all $r \in \mathscr{R}$. An equilibrated force F is of the form $F = \pi^*(F_0)$ for some $F_0 \in (\mathbf{W}/\mathscr{R})^*$. Furthermore, π^* is norm preserving, i.e.,

$$\|\pi^*(F_0)\| = \|F_0\|, \tag{2.17}$$

so one can usually identify an equilibrated F with F_0.

2.2. Generalized stress concentration factors and norms of trace mappings

The central mathematical object that we find suitable for formulating the continuum mechanics problem, particularly, those notions related to stress concentration is $(LD(\Omega)/\mathscr{R})^*$—the dual to the space of LD-distortions. Specifically, as described below and in further detail in [3], elements of this space may be represented by essentially bounded stress fields and on the other hand, the dual of the trace mapping associates an element of $(LD(\Omega)/\mathscr{R})^*$ with any equilibrated boundary traction field.

Consider the composite mapping $\pi \circ \gamma : LD(\Omega) \to L^1(\partial\Omega, \mathbb{R}^3)/\mathscr{R}$. It is noted that for any $r \in \mathscr{R}$, $\gamma(r)$ is a rigid motion on $\partial\Omega$, hence, $\pi \circ \gamma(w + r) = \pi \circ \gamma(w) + \pi \circ \gamma(r) = \pi \circ \gamma(w)$. Thus, we have a well defined mapping

$$\gamma/\mathscr{R} : LD(\Omega)/\mathscr{R} \longrightarrow L^1(\partial\Omega, \mathbb{R}^3)/\mathscr{R}, \tag{2.18}$$

given by $\gamma/\mathscr{R}(\chi) = \pi \circ \gamma(w)$, for some $w \in \chi$. Clearly,

$$\pi \circ \gamma = (\gamma/\mathscr{R}) \circ \pi. \tag{2.19}$$

In addition,

$$\|\gamma(w) + r\|_1 = \|\gamma(w + r)\|_1 \leqslant \|\gamma\| \|w + r\|. \tag{2.20}$$

Hence,

$$\|[\gamma(w)]\| = \inf_{r\in\mathscr{R}} \|\gamma(w) + r\|_1 \leqslant \|\gamma\| \inf_{r\in\mathscr{R}} \|w + r\| = \|\gamma\| \|[w]\| \tag{2.21}$$

and we conclude that $\gamma/\mathscr{R}$ is indeed bounded and

$$\|\gamma/\mathscr{R}\| \leqslant \|\gamma\|. \tag{2.22}$$

The dual mapping $\gamma^* : L^\infty(\partial\Omega, \mathbb{R}^3) \to (LD(\Omega))^*$ may now be applied to traction fields and $(\gamma/\mathscr{R})^* : (L^1(\partial\Omega, \mathbb{R}^3)/\mathscr{R})^* \to (LD(\Omega)/\mathscr{R})^*$ may be applied to equilibrated boundary traction fields to give LD-forces and equilibrated LD-forces respectively. Clearly,

$$\pi^* \circ (\gamma/\mathscr{R})^* = \gamma^* \circ \pi^*. \tag{2.23}$$

Proposition 2.3. *The mappings γ^* and $(\gamma/\mathscr{R})^*$ are injective.*

Proof. The mapping γ^* is injective because γ is continuous and surjective. As the quotient space projection $\pi : L^1(\partial\Omega, \mathbb{R}^3) \to L^1(\partial\Omega, \mathbb{R}^3)/\mathscr{R}$ is also continuous and surjective, the same argument applies to $\gamma/\mathscr{R}$. □

Remark 2.4. Henceforth, we will use the equivalent norm $\|\cdot\|_\varepsilon$ as in (2.9) on $LD(\Omega)/\mathscr{R}$. We will also use the equivalent norm $\|\cdot\|_\oplus$ on $LD(\Omega)$ as in (2.15). This implies that the quotient norm on $LD(\Omega)/\mathscr{R}$ is actually equal (not only equivalent) to the norm induced by the strain. That is,

$$\|[w]\| = \|[w]\|_\varepsilon = \|\varepsilon(w)\|_1. \tag{2.24}$$

It is noted that the last equality, to be used below frequently, is independent of the choice of a particular projection $\pi_{\mathscr{R}}: LD(\Omega) \to \mathscr{R}$.

Proposition 2.5. *Any $T \in (LD(\Omega)/\mathscr{R})^*$ is represented by some symmetric stress field $\sigma \in L^\infty(\Omega, \mathbb{R}^6)^*$, in the form*

$$T = \varepsilon^*(\sigma), \tag{2.25}$$

where $\varepsilon^*: L^\infty(\Omega, \mathbb{R}^6) \to (LD(\Omega)/\mathscr{R})^*$ is the dual mapping to

$$\varepsilon: LD(\Omega)/\mathscr{R} \longrightarrow L^1(\Omega, \mathbb{R}^6). \tag{2.26}$$

In addition, for the dual norm $\|\cdot\|^\varepsilon$ on $(LD(\Omega)/\mathscr{R})^*$, we have

$$\|T\|^\varepsilon = \inf_{\sigma,\, T=\varepsilon^*(\sigma)} \|\sigma\|_\infty, \tag{2.27}$$

and the infimum is attained for some $\hat{\sigma} \in L^\infty(\Omega, \mathbb{R}^6)$, i.e.,

$$\|T\|^\varepsilon = \|\hat{\sigma}\|_\infty. \tag{2.28}$$

Proof. Using the duality $L^1(\Omega, \mathbb{R}^6)^* = L^\infty(\Omega, \mathbb{R}^6)$, the assertion follows from the fact that $\varepsilon: LD(\Omega)/\mathscr{R} \to L^1(\Omega, \mathbb{R}^6)$ is a norm-preserving (by our choice of norm as in Remark 2.4), linear injection and using the Hahn–Banach theorem (see [3] for the details). $\square$

Corollary 2.6. *Let $t \in L^\infty(\partial\Omega, \mathbb{R}^3)$ be any equilibrated traction field so there is a $t_0 \in (L^1(\partial\Omega, \mathbb{R}^3)/\mathscr{R})^*$ such that $t = \pi^*(t_0)$. Then, there exists some stress field $\sigma \in L^\infty(\Omega, \mathbb{R}^6)$ such that*

$$(\gamma/\mathscr{R})^*(t_0) = \varepsilon^*(\sigma), \quad \textit{and} \quad \gamma^*(t) = \pi^* \circ \varepsilon^*(\sigma). \tag{2.29}$$

In addition,

$$\|(\gamma/\mathscr{R})^*(t_0)\| = \inf_{\varepsilon^*(\sigma)=(\gamma/\mathscr{R})^*(t_0)} \|\sigma\|_\infty, \tag{2.30}$$

and

$$\|\gamma^*(t)\| = \inf_{\pi^*\circ\varepsilon^*(\sigma)=\gamma^*(t)} \|\sigma\|_\infty. \tag{2.31}$$

Proof. We make repetitive use of (2.23) and (2.17) and Proposition 2.5. For example,

$$\begin{aligned}
\|\gamma^*(t)\| &= \|\gamma^* \circ \pi^*(t_0)\| \\
&= \|\pi^* \circ (\gamma/\mathscr{R})^*(t_0)\| \\
&= \|(\gamma/\mathscr{R})^*(t_0)\| \\
&= \inf_{\pi^*\circ\varepsilon^*(\sigma)=\pi^*\circ(\gamma/\mathscr{R})^*(t_0)} \|\sigma\|_\infty \\
&= \inf_{\pi^*\circ\varepsilon^*(\sigma)=\gamma^*\circ\pi^*(t_0)} \|\sigma\|_\infty \\
&= \inf_{\pi^*\circ\varepsilon^*(\sigma)=\gamma^*(t)} \|\sigma\|_\infty. \quad \square
\end{aligned} \tag{2.32}$$

Remark 2.7. The conditions (2.29) are equivalent to the principle of virtual work – a weak form of the equations of equilibrium – of continuum mechanics (as it is assumed throughout that the body forces vanish). For example, $\gamma^*(t) = \pi^* \circ \varepsilon^*(\sigma)$, implies

$$\gamma^*(t)(w) = (\pi^* \circ \varepsilon^*)(\sigma)(w), \tag{2.33}$$

so that for any $w \in LD(\Omega)$,

$$t(\gamma(w)) = \sigma(\varepsilon \circ \pi(w)) = \sigma(\varepsilon(w)). \tag{2.34}$$

Hence, for a vector field w that is the restriction of a differentiable field on $\overline{\Omega}$,

$$\int_{\partial\Omega} t_i w_i = \int_{\Omega} \sigma_{ij}\varepsilon(w)_{ij}. \tag{2.35}$$

Theorem 2.8. *Let,*

$$\|\gamma/\mathscr{R}\| = \sup_{\chi\in LD(\Omega)/\mathscr{R}} \frac{\|(\gamma/\mathscr{R})(\chi)\|}{\|\chi\|}, \tag{2.36}$$

be the norm of the trace mapping for distortions. Then, the generalized stress concentration factor K for the boundary traction problem, satisfies

$$K = \|\gamma/\mathscr{R}\|. \tag{2.37}$$

Specifically,

$$K = \sup_{w\in C^\infty(\overline{\Omega})} \frac{\inf_{r\in\mathscr{R}} \|w|_{\partial\Omega} + r\|_1}{\|\varepsilon(w)\|_1}. \tag{2.38}$$

Proof. We have the standard

$$\|\gamma/\mathscr{R}\| = \|(\gamma/\mathscr{R})^*\|. \tag{2.39}$$

However,

$$\|(\gamma/\mathscr{R})^*\| = \sup_{t_0} \frac{\|(\gamma/\mathscr{R})^*(t_0)\|^{\varepsilon}}{\|t_0\|} \tag{2.40}$$

$$= \sup_{t_0\in((\partial\Omega,\mathbb{R}^3)/\mathscr{R})^*} \left\{ \frac{1}{\|t_0\|} \inf_{\varepsilon^*(\sigma)=(\gamma/\mathscr{R})^*(t_0)} \|\sigma\|_\infty \right\} \tag{2.41}$$

$$= \sup_{t\in\mathrm{Image}\,\pi^*} \left\{ \frac{1}{\|t\|_\infty} \inf_{\pi^*\circ\varepsilon^*(\sigma)=\gamma^*(t)} \|\sigma\|_\infty \right\}, \tag{2.42}$$

where we used Proposition 2.5 and Corollary 2.6. The condition $\pi^* \circ \varepsilon^*(\sigma) = \gamma^*(t)$ is equivalent to the condition that the stress field σ is in equilibrium with t, i.e., $\sigma \in \Sigma_t$, by Remark 2.7. The expression (2.38) simply uses the definition of $\|\gamma/\mathscr{R}\|$ and the fact that $C^\infty(\overline{\Omega})$ is dense in $LD(\Omega)$. $\square$

2.3. The scalar case and the trace mapping on the Sobolev space $W_1^1(\Omega)$

The previous discussion is simplified considerably if we consider scalar fields $\varphi \in W_1^1(\Omega)$ instead of the vector fields $w \in LD(\Omega)$. In this case the boundary data is also a scalar field, the analog of a stress is a vector field, and the vector space $\mathscr{R}$ of rigid motions is replaced by the real numbers.

Consider a sourceless vector field σ on Ω that satisfies boundary conditions for its boundary flux, i.e.,

$$\sigma_{i,i} = 0, \quad \text{in } \Omega, \tag{2.43}$$

$$\sigma_i \nu_i = t, \quad \text{on } \partial\Omega, \tag{2.44}$$

for some given essentially bounded $t: \partial\Omega \to \mathbb{R}$. Physically, σ may be thought of as a material flow field (say for an incompressible flow) for a given flux density t on the boundary. Alternatively, σ may be thought of as a heat flow field where there are no heat sources in Ω so t is the given heat flux on the boundary; or σ may be the electric displacement field and t is the charge density on the boundary.

The weak formulation of the problem is

$$\int_\Omega \sigma_i \varphi_{,i} = \int_{\partial\Omega} t\varphi. \tag{2.45}$$

(The test function φ may be thought of as a potential field in the electrostatic example or as the reciprocal of the temperature in the heat transfer example.)

The analog of the stress concentration factor is then

$$K = \sup_{t\in L^\infty(\partial\Omega)} K_t = \sup_{t\in L^\infty(\partial\Omega)} \inf_{\sigma\in\Sigma_t} \frac{\|\sigma\|_{\infty,\overline{\Omega}}}{\|t\|_{\infty,\partial\Omega}} = \sup_{t\in L^\infty(\partial\Omega)} \inf_{\sigma\in\Sigma_t} \frac{\operatorname{ess\,sup}_{x\in\Omega} |\sigma(x)|}{\operatorname{ess\,sup}_{y\in\partial\Omega} |t(y)|}. \tag{2.46}$$

We will refer to K as the *generalized field concentration factor*. Thus for example, for the interpretation of σ as a flow field, for a given flux density t, K_t will be the smallest ratio between the maximal magnitude of the velocity and the maximal value of the given boundary flux.

Thus, φ may be regarded as an element of the Sobolev space $W_1^1(\Omega)$ so

$$\|\varphi\|_{W_1^1} = \|\varphi\|_1 + \|\nabla\varphi\|_1. \tag{2.47}$$

On the Sobolev space, the trace mapping

$$\gamma\colon W_1^1(\Omega) \to L^1(\partial\Omega) \tag{2.48}$$

is well defined as expected (see [9]). Our assumption that there are no sources in Ω implies that $\int_{\partial\Omega} t = 0$ which is equivalent to considering $W_1^1(\Omega)/\mathbb{R}$ and $\gamma/\mathbb{R}$. Thus, with the obvious adaptation of the norm on $W_1^1(\Omega)$ so it is identical to the one on $\mathbb{R} \oplus (W_1^1(\Omega)/\mathscr{R})$, the analog of Theorem 2.8 will be for the scalar case

$$K = \|\gamma/\mathbb{R}\|, \quad \gamma\colon W_1^1(\Omega) \longrightarrow L^1(\partial\Omega). \tag{2.49}$$

3. Bounds on the W_1^1-trace operator

3.1. The bounds obtained using normal vector fields

In this section we consider bounds for the trace operator

$$\gamma\colon W_1^1(\Omega) \longrightarrow L^1(\partial\Omega). \tag{3.1}$$

In particular, we are looking for bounds A and B satisfying

$$\int_{\partial\Omega} |\varphi| \leqslant A \int_\Omega |\nabla\varphi| + B \int_\Omega |\varphi| \tag{3.2}$$

for every $\varphi \in W_1^1(\Omega)$.

Let $\mathfrak{n}$ be any C^1-vector field on $\overline{\Omega}$ and ψ the restriction to $\overline{\Omega}$ of a W_1^1-function defined in an open neighborhood of $\overline{\Omega}$. Then,

$$\int_\Omega \mathfrak{n}_i \psi_{,i} = \int_\Omega (\mathfrak{n}_i\psi)_{,i} - \int_\Omega \mathfrak{n}_{i,i}\psi \tag{3.3}$$

implies using the Gauss–Green theorem that

$$\int_{\partial\Omega} \mathfrak{n}_i \nu_i \psi = \int_\Omega \mathfrak{n}_i \psi_{,i} + \int_\Omega \mathfrak{n}_{i,i}\psi, \tag{3.4}$$

where ν is the outwards pointing unit normal to $\partial\Omega$.

Definition 3.1. The vector field $\mathfrak{n}$ on $\overline{\Omega}$ will be referred to as a normal field if the following conditions hold.

(i) $\mathfrak{n}(y) = \nu(y)$ for all $y \in \partial\Omega$.
(ii) $|\mathfrak{n}|(x) \leqslant 1$ for all $x \in \Omega$, where here and in the rest of this section we use the Euclidean norm for elements of $\mathbb{R}^n$ so $|\mathfrak{n}| = \sqrt{\mathfrak{n}_i \mathfrak{n}_i}$.

The existence of normal vector fields is discussed in some detail below. We will use $\mathfrak{N}(\Omega)$ for the collection of all normal vector fields. For a normal field, Eq. (3.4) assumes the form

$$\int_{\partial\Omega} \psi = \int_{\Omega} \mathfrak{n}_i \psi_{,i} + \int_{\Omega} \mathfrak{n}_{i,i} \psi. \tag{3.5}$$

Given a C^1 mapping φ on $\overline{\Omega}$, the distributional derivatives $|\varphi|_{,i}$ of its absolute value $|\varphi|(x) = |\varphi(x)|$ are clearly integrable and hence, $|\varphi|$ is W_1^1. Rewriting Eq. (3.5) for $\psi = |\varphi|$ we obtain

$$\int_{\partial\Omega} |\varphi| = \int_{\Omega} \mathfrak{n}_i |\varphi|_{,i} + \int_{\Omega} \mathfrak{n}_{i,i} |\varphi|. \tag{3.6}$$

We now estimate each of the integrals on the right hand side.

$$\begin{aligned} \int_{\Omega} \mathfrak{n}_i |\varphi|_{,i} &\leqslant \int_{\Omega} |\mathfrak{n}| \left| |\varphi|_{,i} \right| \\ &\leqslant \int_{\Omega} |\nabla\varphi|, \end{aligned} \tag{3.7}$$

where we used Definition 3.1(ii) and $\left| |\varphi|_{,i} \right| = |\varphi_{,i}| \leqslant |\nabla\varphi|$. Also

$$\int_{\Omega} \mathfrak{n}_{i,i} |\varphi| \leqslant \max_{\overline{\Omega}} \left\{ |\mathfrak{n}_{i,i}| \right\} \int_{\Omega} |\varphi|. \tag{3.8}$$

It follows that

$$\int_{\partial\Omega} |\varphi| \leqslant \int_{\Omega} |\nabla\varphi| + \max_{x \in \overline{\Omega}} \left\{ |\mathfrak{n}_{i,i}(x)| \right\} \int_{\Omega} |\varphi|. \tag{3.9}$$

This inequality is clearly exact and recalling the definition of normal vector fields we have

Theorem 3.2. *Let Ω be a bounded open set in $\mathbb{R}^n$ having a C^2-boundary and set*

$$B(\Omega) = \inf_{\mathfrak{n} \in \mathfrak{N}(\Omega)} \left\{ \max_{x \in \overline{\Omega}} \left\{ |\mathfrak{n}_{i,i}(x)| \right\} \right\} = \inf_{\mathfrak{n} \in \mathfrak{N}(\Omega)} \left\{ \|\mathfrak{n}_{i,i}\|_{\infty,\overline{\Omega}} \right\}. \tag{3.10}$$

Then, the following exact inequality holds

$$\|\varphi\|_{1,\partial\Omega} = \int_{\partial\Omega} |\varphi| \leqslant \int_{\Omega} |\nabla\varphi| + B(\Omega) \int_{\Omega} |\varphi| = \|\nabla\varphi\|_1 + B(\Omega)\|\varphi\|_1. \tag{3.11}$$

3.2. Estimation using harmonic normal fields

We now consider the existence of normal vector fields. By the assumption that $\partial\Omega$ is C^2, ν is a C^1-vector field on $\partial\Omega$ and we can extend it to a vector field $\mathfrak{n}$ with $|\mathfrak{n}(x)| \leqslant 1$ for all $x \in \overline{\Omega}$ (see for example Theorem 3.6.2 in [10]). Furthermore, we can require that the extension is harmonic in the following sense. For a vector field $\mathfrak{n}$, we use $\Delta\mathfrak{n}$ for the vector field $(\Delta\mathfrak{n})_j = (\mathfrak{n}_j)_{,ii}$. The field $\mathfrak{n}$ is *harmonic* in Ω if

$$\Delta\mathfrak{n} = 0, \quad \text{for all } x \in \Omega. \tag{3.12}$$

Theorem 3.3. *Let Ω be a bounded open subset of $\mathbb{R}^n$ such that $\partial\Omega$ is C^2. Then, there exists a unique normal vector field $\mathfrak{n}_0 \in \mathfrak{N}(\Omega)$ which is harmonic. In addition, for the supremum of the divergence, $\nabla \cdot \mathfrak{n}_0 = \mathfrak{n}_{0i,i}$, we have*

$$\|\nabla \cdot \mathfrak{n}_0\|_{\infty,\overline{\Omega}} = \|\nabla \cdot \mathfrak{n}_0\|_{\infty,\partial\Omega}. \tag{3.13}$$

Proof. For any fixed j, we have a classical Dirichlet problem

$$\Delta \mathfrak{n}_j = 0 \quad \text{in } \Omega, \qquad \mathfrak{n}_j = \nu_j \quad \text{on } \partial\Omega. \tag{3.14}$$

Given our smoothness assumption on $\partial\Omega$, there is a unique solution $\mathfrak{n}_{0j}$ to each such boundary value problem and we obtain the harmonic vector field $\mathfrak{n}_0$.

For $|\mathfrak{n}_0|^2 = \mathfrak{n}_{0j}\mathfrak{n}_{0j}$ we have,

$$\begin{aligned} \Delta(\mathfrak{n}_{0j}\mathfrak{n}_{0j}) &= (\mathfrak{n}_{0j}\mathfrak{n}_{0j})_{,ii} \\ &= (2\mathfrak{n}_{0j,i}\mathfrak{n}_{0j})_{,i} \\ &= 2\mathfrak{n}_{0j,ii}\mathfrak{n}_{0j} + 2\mathfrak{n}_{0j,i}\mathfrak{n}_{0j,i}. \end{aligned} \tag{3.15}$$

In the last line, the first term vanishes because $\mathfrak{n}_{0j}$ is harmonic and hence $\Delta(|\mathfrak{n}_0|^2) \geqslant 0$. We conclude that $|\mathfrak{n}_0|^2$ is subharmonic in Ω. By the maximum principle for subharmonic functions

$$\max_{x\in\overline{\Omega}} |\mathfrak{n}_0(x)|^2 = \max_{y\in\partial\Omega} |\mathfrak{n}_0(y)|^2 = \max_{y\in\partial\Omega} |\nu(y)|^2 = 1. \tag{3.16}$$

Thus, in addition to the boundary conditions, $\mathfrak{n}_0$ satisfies the condition 3.1(ii), and so $\mathfrak{n}_0 \in \mathfrak{N}(\Omega)$.

Next, we note that for the harmonic $\mathfrak{n}_0$,

$$\begin{aligned} \Delta(\nabla \cdot \mathfrak{n}_0) &= (\mathfrak{n}_{0j,j})_{,ii} \\ &= (\mathfrak{n}_{0j,ii})_{,j} \\ &= 0. \end{aligned} \tag{3.17}$$

Thus, $\nabla \cdot \mathfrak{n}_0$ is also harmonic in Ω so by the maximum principle Eq. (3.13) holds. □

It turns out that the harmonic vector field of Theorem 3.3 plays an important role in the computation of $B(\Omega)$, the second constant of the Sobolev $W_1^1(\Omega)$ trace inequality. Continuing to use $\mathfrak{n}_0$ for the unique harmonic normal vector field we have

Theorem 3.4. *The Sobolev constant $B(\Omega)$ is given by*

$$B(\Omega) = \inf_{\mathfrak{n}\in\mathfrak{N}(\Omega)} \|\nabla \cdot \mathfrak{n}\|_{\infty,\overline{\Omega}} = \|\nabla \cdot \mathfrak{n}_0\|_{\infty,\partial\Omega}. \tag{3.18}$$

Proof. Let $(\mathfrak{n}_m)$, $m \in \mathbb{N}$, be a sequence of normal vector fields such that

$$\lim_{m\to\infty} \|\nabla \cdot \mathfrak{n}_m\| = B(\Omega). \tag{3.19}$$

Using normal tubular neighborhoods (e.g., [11, p. 110]), there is a $\delta > 0$ such that we can parameterize an open neighborhood V of $\partial\Omega$ in Ω by

$$(y, z) \in \partial\Omega \times [0, \delta), \tag{3.20}$$

where for each $x \in V$, $y(x)$ is a unique point on the boundary such that x is on the line through y which is normal to the boundary, and z is the distance to the boundary along the normal line where x is situated. For each m, let V_m be the open set

$$V_m = \left\{x \in \overline{\Omega} : y(x) \in \partial\Omega, z(x) < \delta/m\right\}, \tag{3.21}$$

and let $\Omega_m = \overline{\Omega} - \overline{V}_m$ so

$$\partial\Omega_m = \left\{x \in \overline{\Omega} : y(x) \in \partial\Omega, z(x) = \delta/m\right\}. \tag{3.22}$$

Now for each m we construct the harmonic lifting (cf. [12, p. 24]) $\overline{\mathfrak{n}}_m$ of $\mathfrak{n}_m$ as follows. Let $\mathfrak{n}_{0m}$ be the solution of the Dirichlet problem on Ω_m with the boundary conditions $\mathfrak{n}_{0m}(x) = \mathfrak{n}_m(x)$ for $x \in \partial\Omega_m$. Set

$$\overline{\mathfrak{n}}_m(x) = \begin{cases} \mathfrak{n}_{0m}(x), & \text{for } x \in \Omega_m, \\ \mathfrak{n}_m(x), & \text{for } x \in \overline{V}_m. \end{cases} \tag{3.23}$$

By the maximum principle,

$$\|\overline{\mathfrak{n}}_{mi}\|_{\infty,\overline{\Omega}} \leqslant \|\mathfrak{n}_{mi}\|_{\infty,\overline{\Omega}} \quad \text{and} \quad \|\nabla \cdot \overline{\mathfrak{n}}_m\|_{\infty,\overline{\Omega}} \leqslant \|\nabla \cdot \mathfrak{n}_m\|_{\infty,\overline{\Omega}}, \tag{3.24}$$

so

$$\lim_{m\to\infty} \|\nabla \cdot \overline{\mathfrak{n}}_m\|_{\infty,\overline{\Omega}} = B(\Omega). \tag{3.25}$$

In addition, as the various $\|\mathfrak{n}_{mi}\|_{\infty,\overline{\Omega}}$ are bounded by 1, the same applies to $\overline{\mathfrak{n}}_{mi}$ and the sequence $\overline{\mathfrak{n}}_{mi}$ is uniformly bounded. Thus, (using a standard normal family argument) by Ascoli's theorem, it has a subsequence that converges uniformly to a limit continuous normal field $\overline{\mathfrak{n}}$. On any compact subset of Ω this gives a uniformly convergent sequence of harmonic functions whose limit is then also harmonic. Thus, $\overline{\mathfrak{n}}$ is harmonic. Also, the limit $\overline{\mathfrak{n}}$ satisfies the conditions of Definition 3.1 and so it is a normal vector field. Finally, by the uniqueness of the solution to the Dirichlet problem $\overline{\mathfrak{n}} = \mathfrak{n}_0$. □

Remark 3.5. Within the framework of the AB-program in geometric analysis, [13], Motron proves in [8], using different methods, the following two theorems for bounds on the trace mapping on $W_1^1(\Omega)$.

(1) For any $\varepsilon > 0$, there exists a B_ε such that for any $\varphi \in W_1^1(\Omega)$,

$$\int_{\partial\Omega} |\varphi| \leqslant (1+\varepsilon)\int_\Omega |\nabla\varphi| + B_\varepsilon \int_\Omega |\varphi|. \tag{3.26}$$

(2) Assuming that Ω is a connected bounded open subset of $\mathbb{R}^n$ whose boundary is piecewise C^1, there exists $A > 0$ such that for any $\varphi \in W_1^1(\Omega)$,

$$\int_{\partial\Omega} |\varphi| \leqslant A\int_\Omega |\nabla\varphi| + \frac{|\partial\Omega|}{|\Omega|} \int_\Omega |\varphi|. \tag{3.27}$$

We note that even if we had the solution to the AB-program, we would not have a concrete bound on $\|\gamma\|$. What we sought were simultaneous bounds on both A and B.

In addition, for a normal vector field

$$\sup_{x\in\overline{\Omega}} |\nabla \cdot \mathfrak{n}||\Omega| \geqslant \int_\Omega \nabla \cdot \mathfrak{n} = \int_{\partial\Omega} \mathfrak{n} \cdot \nu = |\partial\Omega|, \tag{3.28}$$

so

$$\inf_{\mathfrak{n}\in\mathfrak{N}} \|\nabla \cdot \mathfrak{n}\|_{\infty,\overline{\Omega}} \geqslant \frac{|\partial\Omega|}{|\Omega|}. \tag{3.29}$$

However, the inequality is not exact and equality is not attainable even for the harmonic normal vector field. (Think of two circles connected by a narrow neck of width t. Across the narrow neck, $\nabla \cdot \mathfrak{n}$ has to be of order $1/t$ in order to satisfy the boundary conditions. The values of $|\Omega|$ and $|\partial\Omega|$ are not significantly different from those of the two circles.) Thus, Motron's bound B is smaller than the value obtained here. On the other hand, the bounds A and B we obtain are exact in the sense that you cannot lower B without increasing A.

4. Norms of symmetric tensors

As noted earlier, the value of the stress concentration factor depends on the norm we choose to use on the space of stress matrices. Thus, for the sake of completeness, we review below some elementary properties of norms of symmetric matrices on $\mathbb{R}^n$. We will denote the norm of a matrix T by $|T|$ (reserving $\|\cdot\|$ for norms on function spaces).

4.1. Operator norms

In general, for a linear mapping $T: \mathbf{V} \to \mathbf{U}$ between normed spaces, the operator norm of T is defined by

$$|T|_o = \sup_v \frac{|T(v)|}{|v|}, \quad v \neq 0. \tag{4.1}$$

The operator p-norm, $|T|_{op}$, $1 \leqslant p \leqslant \infty$, on the space of matrices is defined as the operator norm for the case where the p-norm is used on both $\mathbf{V} = \mathbb{R}^m$ and $\mathbf{U} = \mathbb{R}^n$. By the compactness of the unit ball in $\mathbb{R}^m$, the supremum is attainable and

$$|T|_{op} = \max_{|v|=1} |T(v)|_p. \tag{4.2}$$

In case $T: \mathbf{W} \to \mathbf{W}$ is a linear transformation defined on the inner product space $\mathbb{R}^n$ equipped with the Euclidean 2-norm, $|T|_{o2}$ may be calculated by

$$|T|_{o2} = \sup_{v,v' \in \mathbf{W}} \frac{|T(v) \cdot v'|}{|v||v'|}, \quad v, v' \neq 0. \tag{4.3}$$

The following relations hold for symmetric matrices on $\mathbb{R}^n$.

$$|T|_{o1} = |T|_{o\infty} = \max_i \sum_j |T_{ij}|, \tag{4.4}$$

$$|T|_{o2} = \max\{|\lambda_1|, \ldots, |\lambda_n|\}, \tag{4.5}$$

where $\lambda_1, \ldots, \lambda_n$ are the (real) eigenvalues of T. The norm $|\cdot|_{o2}$ is usually referred to as the spectral radius norm.

4.2. Vector norms

We will also regard symmetric matrices as vectors in $\mathbb{R}^{n(n+1)/2}$ and use the p-norm for them. Thus,

$$|T|_p = \left(\sum_{i,j} |T_{ij}|^p \right)^{1/p}. \tag{4.6}$$

In particular,

$$|T|_1 = \sum_{i,j} |T_{ij}|, \tag{4.7}$$

$$|T|_\infty = \sup_{i,j} |T_{ij}|, \tag{4.8}$$

$$|T|_2 = \sqrt{T_{ij} T_{ji}} = \left(\sum \lambda_i^2 \right)^{1/2}. \tag{4.9}$$

We recall that the Frobenius norm of a matrix is $|T|_F = (T_{ij} T_{ij})^{1/2}$ and it is identical to $|\cdot|_2$ for symmetric matrices.

4.3. Dual norms

Being a finite dimensional space we may identify the space of symmetric matrices $\mathbb{R}^{n(n+1)/2}$ with its dual space. Thus, we may regard any symmetric matrix T as a linear functional so that $T(S) = T_{ij} S_{ji}$ and assign to it the dual norm $|T|_{p^*}$

$$|T|_{p^*} = \sup_S \frac{|T(S)|}{|S|_p}, \tag{4.10}$$

where we have the usual $|\cdot|_{p^*} = |\cdot|_q$, for $q = p/(p-1)$.

Dual norms may be used also for the operator norms. In particular, we note that

$$|T|_{o2^*} = \sum_i |\lambda_i|. \tag{4.11}$$

In closing this short review, it is noted that the norms containing the index 2 are associated with the Euclidean norm for vectors and may be expressed in terms of the eigenvalues. These norms are invariant under orthogonal transformations of coordinates.

4.4. The equivalence constants

Since all the norms listed above are equivalent, for each pair of norms $|\cdot|_a$ and $|\cdot|_b$, there is a finite positive number

$$K_b^a = \sup_T \frac{|T|_a}{|T|_b}. \tag{4.12}$$

In particular, the following exact relations hold:

$$\frac{1}{\sqrt{n}} \leqslant \frac{|T|_{o2}}{|T|_{o1}} \leqslant \sqrt{n}, \qquad 1 \leqslant \frac{|T|_2}{|T|_{o2}} \leqslant \sqrt{n}, \qquad 1 \leqslant \frac{|T|_{o2}}{|T|_\infty} \leqslant n, \tag{4.13}$$

$$1 \leqslant \frac{|T|_{o2^*}}{|T|_{o2}} \leqslant n, \qquad 1 \leqslant \frac{|T|_{o2^*}}{|T|_2} \leqslant \sqrt{n}. \tag{4.14}$$

Thus, for example, $K_2^{o2} = 1$.

In the sequel, we will use $|\sigma(x)|$ to denote the norm of the value of a symmetric tensor σ at $x \in \Omega$ using a generic norm on the space of matrices.

5. Optimal boundary conditions for stresses

Consider the following problem: given a unit vector ν in a Euclidean 3-dimensional space and a unit vector t, find a symmetric matrix σ such that

(i) $\sigma(\nu) = t$—the compatibility condition;
(ii) $|\sigma| = \inf\{|T|, T(\nu) = t, T = T^{\mathrm{T}}\}$, i.e., σ is the optimal symmetric matrix that satisfies condition (i).

The problem has an obvious mechanical interpretation. If ν denotes the normal to the boundary at some given point, and t denotes the value of the surface traction field at that point, then, $\sigma(\nu) = t$ is the boundary condition for the stress field σ. Thus, a matrix σ satisfying the conditions above is the optimal stress matrix that will satisfy the boundary condition. Obviously, the normalization condition on t causes no loss of generality.

Let $t_n = (t \cdot \nu)\nu = \nu \otimes \nu(t)$ be the normal component of t and let $t_t = \nu \times (t \times \nu)$ be the tangent component of t. Thus, denoting the angle between ν and t by θ, $|t_n| = \cos\theta$ and $|t_t| = \sin\theta$. We choose a basis $\{f_j\}$ where $f_1 = \nu$, f_2 is a unit vector in the direction of t_t and f_3 completes the other two to form a right-hand oriented orthonormal basis. In this basis, the matrix of σ satisfying the condition $\sigma(\nu) = t$ has to satisfy $\sigma_{11} = \cos\theta$, $\sigma_{12} = \sigma_{21} = \sin\theta$, and $\sigma_{13} = \sigma_{31} = 0$. The rest of the components cannot be determined by the compatibility condition above and should be determined by the requirement for minimal norm of σ. (In the case where ν and t are parallel, one can take any orthonormal basis containing ν.)

5.1. Optimal boundary conditions relative to the $|\cdot|_\infty$-norm

We wish to regard σ as an element of the dual space of symmetric matrices. Then, using the basis f_i as above, the compatibility condition implies that we have a linear functional σ_0 defined on the subspace $\mathbf{V}$ of symmetric matrices containing elements of the form

$$[\varepsilon] = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & 0 \\ \varepsilon_{12} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{5.1}$$

The functional σ_0 acts on elements of $\mathbf{V}$ by

$$\sigma_0([\varepsilon]) = \varepsilon_{11}\cos\theta + 2\varepsilon_{12}\sin\theta. \tag{5.2}$$

Thus, on this subspace

$$\sup\{|\sigma_0(\varepsilon)|;\, \varepsilon \in \mathbf{V},\ |\varepsilon|_1 = 1\} = \max\{|\cos\theta|, |\sin\theta|\}. \tag{5.3}$$

The extension

$$[\sigma] = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ \sin\theta & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{5.4}$$

of σ_0 to the space of all symmetric matrices, has the same norm and as such it provides the optimal boundary condition. It is noted that while the development depends on the basis chosen, the optimal norm depends only on the angle θ, an invariant quantity.

5.2. *Optimal boundary conditions relative to the* $|\cdot|_2$*-norm*

If one uses $|\sigma| = |\sigma|_2$ induced by the inner product in the space of symmetric matrices as in the previous section, the optimal matrix can be obtained by orthogonality conditions of the optimal stress to basis vectors for the matrices that correspond to the undetermined components of the stress. This implies that all the undetermined components should vanish. Thus, the optimal stress is given in the $\{f_i\}$-basis by Eq. (5.4) also, and

$$|\sigma|_2 = \sqrt{1 + \sin^2\theta}. \tag{5.5}$$

While the optimal σ for the $|\cdot|_\infty$-norm depended on the basis $\{f_i\}$ chosen, the construction here is rotation-invariant.

5.3. *The case where* $t = e_k$

We now consider the special case where $t = e_k$, where e_k, $k = 1, 2, 3$ is a base vector. In this case, $\cos\theta = \nu \cdot e_k = \nu_k$ and $\sin\theta = \sqrt{1 - \nu_k^2}$. Using

$$\sigma = \cos\theta\, \nu \otimes \nu + \sin\theta(\nu \otimes f_2 + f_2 \otimes \nu), \tag{5.6}$$

we may write the matrix for σ relative to the $\{e_i\}$-basis. We first note that

$$f_2 = \frac{e_k - \cos\theta\, \nu}{\sin\theta} = \frac{e_k - \nu_k \nu}{\sqrt{1 - \nu_k^2}}, \tag{5.7}$$

thus,

$$\sigma = \nu_k \nu \otimes \nu + \nu \otimes (e_k - \nu_k \nu) + (e_k - \nu_k \nu) \otimes \nu. \tag{5.8}$$

Rearranging the terms we conclude that for the case $t = e_k$, the optimal boundary conditions for the stress are

$$\sigma = -\nu_k \nu \otimes \nu + (\nu \otimes e_k + e_k \otimes \nu), \tag{5.9}$$

and the optimal values are

$$|\sigma|_2 = \sqrt{2 - \nu_k^2},$$
$$|\sigma|_\infty = \max\left\{|\nu_k|, \sqrt{1 - \nu_k^2}\right\}, \quad \text{for the natural basis } \{f_i\}.$$

5.4. Example: The 2-dimensional case for the $|\cdot|_{o2}$-spectral radius norm

Using the same notation as above and using the coordinate system in the plane where the x and y axes are along f_1 and f_2, respectively, we are looking for a 2×2 symmetric matrix that will satisfy the condition $t = \sigma(\nu)$ of least spectral radius, i.e., minimizes $\max\{|\lambda_1|, |\lambda_2|\}$. The condition $\sigma(\nu) = t$ implies that $\sigma_{xx} = \cos\theta$ and $\sigma_{xy} = \sin\theta$. Thus, to determine σ completely, one has to determine the single number σ_{yy}.

In two dimensions we have the explicit expression for the eigenvalues as

$$\lambda_{1,2} = \frac{\sigma_{xx} + \sigma_{yy}}{2} \pm \sqrt{\left(\frac{\sigma_{xx} - \sigma_{yy}}{2}\right)^2 + {\sigma_{xy}}^2}. \tag{5.10}$$

Setting

$$a = \frac{\cos\theta}{2} = \frac{\sigma_{xx}}{2}, \qquad z = \frac{\sigma_{yy}}{2}, \quad \text{so } \sigma_{xy}^2 = \sin^2\theta = 4 - a^2, \tag{5.11}$$

we have

$$\lambda_{1,2} = a + z \pm \sqrt{1 - 3a^2 - 2az + z^2} = a + z \pm \sqrt{(z-a)^2 + 1 - 4a^2}. \tag{5.12}$$

Minimizing $|\lambda_{1,2}|$ is like minimizing $\lambda_{1,2}^2$ so we differentiate with respect to z. Thus,

$$\begin{aligned} \frac{\mathrm{d}\lambda_{1,2}^2}{\mathrm{d}z} &= 2\lambda_{1,2}\frac{\mathrm{d}\lambda_{1,2}}{\mathrm{d}z} \\ &= 2\lambda_{1,2}\left(1 \pm \frac{z-a}{\sqrt{(z-a)^2 + 1 - 4a^2}}\right) \end{aligned}$$

that vanishes identically only if $4a^2 = 1$. Hence, for extremum, $\theta = 0$. In general,

$$1 - 4a^2 = 1 - \cos^2\theta > 0, \tag{5.13}$$

which implies that

$$\frac{\mathrm{d}\lambda_{1,2}^2}{\mathrm{d}z} \tag{5.14}$$

has the same sign as the eigenvalue. It follows that $z \neq 0$ can only make $|\lambda_{1,2}|$ larger and the infimum is attained for $z = \sigma_{yy} = 0$.

Remark 5.1. We note that the problem of optimal boundary condition for the stress is a generalization of the requirement in Definition 3.1(i) for the boundary value of a normal vector field with the difference that now we consider matrices rather than vectors. Indeed, $\mathfrak{n} = \nu$ gives the smallest value for $|\mathfrak{n}|_2$, the Euclidean norm of the vector field $\mathfrak{n}$, such that $\mathfrak{n} \cdot \nu = \mathfrak{n}(\nu) = 1$.

5.5. Worst case optimal boundary conditions

By *worst case optimal boundary conditions* we refer to the inclination of t relative to the normal ν for which the norm of the optimal stress attains a maximal value. That is, we are looking for

$$D_{\|} = \sup_t \left\{ \inf_\sigma |\sigma| : \sigma(\nu) = t, |t| = 1 \right\}. \tag{5.15}$$

The number $D_{\|}$ (the subscript $\|$ indicating the particular norm chosen) and the corresponding σ and t depend only on the choice of norms. For example, for the $|\cdot|_2$-norm, $D_2 = \sqrt{2}$ is attained for any unit traction t perpendicular to ν. For the $|\cdot|_\infty$-norm, relative to the boundary natural basis, $D_\infty = 1$ is attained for either traction that is parallel to ν or traction that is perpendicular to it.

5.6. The spaces $\mathfrak{N}^k(\Omega)$ and e_k-optimal tensor fields

For $k = 1, 2, 3$ we now choose a symmetric tensor field $T^k = (T^k_{ij})$ on $\partial\Omega$ that satisfies the following conditions.

(i) If e_k denotes the k-th base vector in $\mathbb{R}^3$, then, $T^k(\nu) = e_k$ so $T^k_{ij}\nu_j = \delta^k_i$.
(ii) Clearly, at each point y on the boundary there is a collection of matrices that satisfy condition (i) above. We choose $T^k(y)$ to be a symmetric matrix that satisfies condition (i) and that has the least norm (of our choice on the space of matrices). Thus,

$$|T^k(y)| = \inf_{S^k}\left\{|S^k| : S^k(\nu(y)) = e_k\right\}, \tag{5.16}$$

where the infimum is taken over all symmetric matrices. Thus, in the terminology of the preceding subsections, T^k is the optimal stress for the e_k as boundary traction and the discussion of Section 5.3 applies.

Clearly, the fields T^k depend on Ω only. They depend continuously on ν so by the assumption that $\partial\Omega$ is C^2 they are continuous. Consider now the worst value of $|T^k(y)|$ on the boundary, i.e.,

$$\sup_{y\in\partial\Omega}\left\{|T^k(y)|\right\} = \sup_{y\in\partial\Omega}\left\{\inf_{S^k}\left\{|S^k| : S^k(\nu(y)) = e_k\right\}\right\}. \tag{5.17}$$

As $\partial\Omega$ is assumed to be smooth, any angle between ν and any fixed vector is attained on the boundary, hence, the worst case optimal boundary conditions are attained on the boundary always. Thus,

$$\sup_{y\in\partial\Omega}\left\{|T^k(y)|\right\} = \sup_{y\in\partial\Omega}\left\{\inf_{S^k}\left\{|S^k|: S^k(\nu(y)) = e_k\right\}\right\} = D_{\|} \tag{5.18}$$

and depends only on the choice of a norm.

We use the notation T^k for a tensor field on the boundary satisfying the two conditions above. Using Whitney's extension theorem (cf. [10, Theorem 3.6.2]), T^k can be extended to symmetric differentiable tensor fields σ^k on $\overline{\Omega}$ that satisfy the following condition

$$\sup_{x\in\overline{\Omega}}\left\{|\sigma^k(x)|\right\} = \sup_{y\in\partial\Omega}\left\{|T^k(y)|\right\} = D_{\|}. \tag{5.19}$$

We denote the class of symmetric C^1-tensor fields σ^k on $\overline{\Omega}$ that satisfy the boundary condition $\sigma^k(y) = T^k(y)$, $y \in \partial\Omega$, and condition (5.19) above by $\mathfrak{N}^k(\Omega)$. We will refer to a tensor field $\sigma^k \in \mathfrak{N}^k(\Omega)$ as an *e_k-optimal tensor field*.

It is quite clear that the foregoing discussion applies in the continuum mechanics context to the optimal stress field for the normalized boundary traction $t = e_k$. If $\sigma^k \in \mathfrak{N}^k(\Omega)$, we have for the stress concentration factor and optimal stress concentration factor,

$$K_{e_k,\sigma^k} = K_{e_k} = D_{\|}. \tag{5.20}$$

Although the total force on the body is not equilibrated, $D_{\|}$, may serve as a bound.

6. Bounds of the $LD(\Omega)$-trace operator

Theorem 6.1. *Let the constants $A(\Omega)$ and $B(\Omega)$ be given by*

$$A(\Omega) = 3D_{\|}, \tag{6.1}$$

$$B(\Omega) = \sum_k \|\nabla\cdot\sigma^k_0\|_{\infty,\partial\Omega}, \tag{6.2}$$

where σ^k_0 is the solution of the Dirichlet problem $\Delta\sigma^k_{ij} = 0$, in Ω, $\sigma^k_{ij} = T^k_{ij}$, on $\partial\Omega$. Then,

$$\|w\|_{1,\partial\Omega} \leqslant A(\Omega)\|\varepsilon(w)\|_1 + B(\Omega)\|w\|_1, \tag{6.3}$$

for all $w \in LD(\Omega)$, is an exact estimate.

The following subsections present the proof. (We will use A and B for $A(\Omega)$ and $B(\Omega)$, respectively, in order to simplify the notation.)

6.1. The principle of virtual work

Let Ω be an open region in $\mathbb{R}^3$ having a smooth boundary, $\sigma = (\sigma_{ij})$ a symmetric smooth tensor field on Ω and $w = (w_i)$ an LD-vector field on Ω. Then,

$$\sigma_{ij} w_{i,j} = (\sigma_{ij} w_i)_{,j} - \sigma_{ij,j} w_i. \tag{6.4}$$

Also, by the symmetry of σ,

$$\sigma_{ij} w_{i,j} = \sigma_{ij} \varepsilon_{ij}. \tag{6.5}$$

Thus, we may write

$$\int_\Omega \sigma_{ij} \varepsilon_{ij} = \int_\Omega (\sigma_{ij} w_i)_{,j} + \int_\Omega \sigma_{ij,j} w_i, \tag{6.6}$$

and using the Green–Gauss theorem on the first term on the right-hand side we obtain

$$\int_\Omega \sigma_{ij} \varepsilon_{ij} = \int_\Omega \sigma_{ij} \frac{1}{2}(w_{i,j} + w_{j,i}) = \int_{\partial\Omega} \sigma_{ij} w_i \nu_j + \int_\Omega \sigma_{ij,j} w_i, \tag{6.7}$$

where ν is the unit outward pointing normal. We will refer to the identity above as the *principle of virtual work*.

6.2. The bounds

We now write the principle of virtual work (6.7) for the vector field $\langle w \rangle = (|w_i|)$. (We reserve the notation $|v|$ for the norm of the vector v in a finite dimensional space.) Thus,

$$\int_\Omega \sigma_{ij} \frac{1}{2}(|w_i|_{,j} + |w_j|_{,i}) = \int_{\partial\Omega} \sigma_{ij} |w_i| \nu_j + \int_\Omega \sigma_{ij,j} |w_i|. \tag{6.8}$$

Let σ^k satisfy $\sigma^k(\nu) = e_k$ on $\partial\Omega$ so $\sigma^k_{ij} \nu_j = \delta^k_i$. Then, the identity above assumes the form

$$\int_{\partial\Omega} |w_k| = \int_\Omega \frac{1}{2}(|w_i|_{,j} + |w_j|_{,i}) \sigma^k_{ij} - \int_\Omega \sigma^k_{ij,j} |w_i|. \tag{6.9}$$

Consider the integrand

$$\text{integrand} = \frac{1}{2}(|w_i|_{,j} + |w_j|_{,i}) \sigma^k_{ij} \tag{6.10}$$

of the first integral on the right. As the expression is invariant under orthogonal transformations, it may be evaluated in the principle coordinate system of the matrix $(|w_i|_{,j} + |w_j|_{,i})/2$ where the off-diagonal elements vanish. Thus, without loss of generality, we may write (we do not use the summation convention here)

$$\begin{aligned}
\text{integrand} &= \sum_i |w_i|_{,i} \sigma^k_{ii} \\
&\leqslant \max_i\{|\sigma^k_{ii}|\} \sum_i |w_i|_{,i} \\
&\leqslant |\sigma^k|_\infty \sum_i |w_{i,i}| \quad (\text{using } |w_i|_{,j} = \operatorname{sign}(w_i) w_{i,j}) \\
&\leqslant |\sigma^k|_\infty |\varepsilon(w)|_1,
\end{aligned}$$

where the equality is clearly attainable. Thus,

$$\begin{aligned}\int_{\partial\Omega} |w_k| &\leqslant \int_{\Omega} |\sigma^k|_\infty |\varepsilon(w)|_1 + \int_{\Omega} |\sigma^k_{ij,j}||w_i| \\ &\leqslant \sup_{x\in\overline{\Omega}} \left\{|\sigma^k(x)|_\infty\right\} \int_{\Omega} |\varepsilon(w)|_1 + \sup_{i,x\in\overline{\Omega}} \{|\sigma^k_{ij,j}(x)|\} \int_{\Omega} \sum_i |w_i|.\end{aligned}$$

As

$$\sup_{i,x\in\overline{\Omega}} |\sigma^k_{ij,j}(x)| = \|\nabla\cdot\sigma^k\|_{\infty,\overline{\Omega}}, \tag{6.11}$$

we have

$$\|w_k\|_{1,\partial\Omega} = \int_{\partial\Omega} |w_k| \leqslant \|\sigma^k\|_{\infty,\overline{\Omega}} \|\varepsilon(w)\|_1 + \|\nabla\cdot\sigma^k\|_{\infty,\overline{\Omega}} \|w\|_1, \tag{6.12}$$

where we use $\|T\|_\infty = \||T|_\infty\|_{L^\infty}$, and $\|T\|_1 = \||T|_1\|_{L^1}$, for the respective norms of a tensor field T.

Adding this equation for $k = 1, 2, 3$, we obtain for the L^1-norm of the restriction of w to the boundary the following bound

$$\|w\|_{1,\partial\Omega} \leqslant \sum_k \|\sigma^k\|_{\infty,\overline{\Omega}} \|\varepsilon(w)\|_1 + \sum_k \|\nabla\cdot\sigma^k\|_{\infty,\overline{\Omega}} \|w\|_1. \tag{6.13}$$

Clearly, for

$$A = \inf_{\sigma^k} \left\{ \sum_k \|\sigma^k\|_{\infty,\overline{\Omega}} : \sigma^k(\nu) = e_k \right\} \tag{6.14}$$

and

$$B = \inf_{\sigma^k} \left\{ \sum_k \|\nabla\cdot\sigma^k\|_{\infty,\overline{\Omega}} : \sigma^k(\nu) = e_k \right\} \tag{6.15}$$

this bound is the tightest and we have

$$\|w\|_{1,\partial\Omega} \leqslant A\|\varepsilon(w)\|_1 + B\|w\|_1, \quad \text{for all } w \in LD(\Omega). \tag{6.16}$$

Since in general

$$\begin{aligned}\sup_{x\in\overline{\Omega}} |\sigma^k(x)|_\infty &\geqslant \sup_{y\in\partial\Omega} |\sigma^k(y)|_\infty \\ &\geqslant \sup_{y\in\partial\Omega} |T^k(y)|_\infty, \quad T^k \text{optimal as in (5.16)} \\ &= D_\parallel\end{aligned}$$

and equality holds for $\sigma^k \in \mathfrak{N}^k(\Omega)$, we conclude that A is attained for fields $\sigma^k \in \mathfrak{N}^k(\Omega)$ and $A = 3D_\parallel$. This proves the first part (Eq. (6.1)) of Theorem 6.1.

6.3. Estimating B

The procedure we use is completely analogous to that of Section 3.2 and the proofs of Theorems 3.3 and 3.4. For a tensor field σ we use $\Delta\sigma$ for the Laplacian $\Delta\sigma_{ij} = \sigma_{ij,ll}$. We say that σ is *harmonic* if $\Delta\sigma = 0$.

Proposition 6.2. *There is a unique harmonic tensor field $\sigma^k \in \mathfrak{N}^k(\Omega)$, i.e., σ^k is e_k-optimal. For the harmonic e_k-optimal σ^k, we have*

$$\|\nabla\cdot\sigma^k\|_{\infty,\overline{\Omega}} = \|\nabla\cdot\sigma^k\|_{\infty,\partial\Omega}. \tag{6.17}$$

Proof. Consider the Dirichlet problem

$$\Delta\sigma^k = 0, \quad \text{in } \Omega, \qquad \sigma^k = T^k, \quad \text{on } \partial\Omega. \tag{6.18}$$

The existence and uniqueness are standard. Let σ^k be harmonic, then, for each component σ^k_{ij} (no sum on repeated indices)

$$\begin{aligned}\Delta((\sigma^k_{ij})^2) &= (\sigma^k_{ij}\sigma^k_{ij})_{,ll}, \\ &= 2(\sigma^k_{ij,l}\sigma^k_{ij})_{,l} \\ &= 2\sigma^k_{ij,l}\sigma^k_{ij,l} + 2\sigma^k_{ij,ll}\sigma^k_{ij} \\ &\geqslant 0.\end{aligned} \tag{6.19}$$

Thus, σ^k_{ij} is subharmonic in Ω. By the maximum principle for subharmonic functions

$$\max_{x\in\overline{\Omega}}(\sigma^k_{ij}(x))^2 = \max_{y\in\partial\Omega}(\sigma^k_{ij}(y))^2 \tag{6.20}$$

and so the analogous property holds for $|\sigma^k_{ij}|$. Thus, using the boundary conditions

$$\|\sigma^k\|_{\infty,\overline{\Omega}} = \max_{i,j,x\in\overline{\Omega}}|\sigma^k_{ij}(x)| = \max_{i,j,y\in\partial\Omega}|\sigma^k_{ij}(y)| = \max_{i,j,y\in\partial\Omega}|T^k_{ij}(y)| = D_\infty. \tag{6.21}$$

Hence, the solution is also in $\mathfrak{N}^k(\Omega)$. Finally,

$$\begin{aligned}\Delta(\nabla\cdot\sigma^k) &= (\sigma^k_{ij,j})_{,ll} \\ &= (\sigma^k_{ij,ll})_{,j} \\ &= 0,\end{aligned} \tag{6.22}$$

so by the maximum principle for the components of $\nabla\cdot\sigma^k$, (6.17) holds. □

Proof of the second part of Theorem 6.1 (Eq. (6.2)). Clearly, as the three σ^k fields are independent, we should look for

$$B_k = \inf_{\sigma^k}\left\{\|\nabla\cdot\sigma^k\|_{\infty,\overline{\Omega}} : \sigma^k(\nu) = e_k\right\}. \tag{6.23}$$

Thus, let (σ^k_m), $m\in\mathbb{N}$, be a sequence of e_k-optimal tensor fields such that

$$\lim_{m\to\infty}\|\nabla\cdot\sigma^k_m\| = B_k. \tag{6.24}$$

The sets V_m and Ω_m may be constructed just as in the proof of Theorem 3.4. Let σ^k_{0m} be the solution of the Dirichlet problem in Ω_m with the boundary condition $\sigma^k_{0m}(x) = \sigma^k_m(x)$, for $x\in\partial\Omega_m$, and define the harmonic lifting accordingly as

$$\overline{\sigma}^k_m(x) = \begin{cases}\sigma^k_{0m}(x), & \text{for } x\in\Omega_m, \\ \sigma^k_m(x), & \text{for } x\in\overline{V}_m.\end{cases} \tag{6.25}$$

By the maximum principle

$$\|\overline{\sigma}^k_{mij}\|_{\infty,\overline{\Omega}} \leqslant \|\sigma^k_{mij}\|_{\infty,\overline{\Omega}} \quad \text{and} \quad \|\nabla\cdot\overline{\sigma}^k_m\|_{\infty,\overline{\Omega}} \leqslant \|\nabla\cdot\sigma^k_m\|_{\infty,\overline{\Omega}}, \tag{6.26}$$

so

$$\lim_{m\to\infty}\|\nabla\cdot\overline{\sigma}^k_m\|_{\infty,\overline{\Omega}} = B_k. \tag{6.27}$$

We apply the normal family argument and uniqueness of solutions as in the proof of Theorem 3.4, to obtain

$$B_k = \|\nabla\cdot\sigma^k_0\|_{\infty,\partial\Omega}, \tag{6.28}$$

where σ^k_0 is the solution of the Dirichlet problem (6.18). Eq. (6.2) now follows from (6.15). □

7. Discussion

We may describe the foregoing analysis in the mechanical context. For a given boundary traction field, we constructed the optimal boundary condition for the stress field. In Section 5, the case where the traction vector was a unit vector was considered, but as the relation between stress and traction is linear, this causes no loss of generality. Thus, one can assign the optimal boundary condition for the stress field for any given boundary traction field.

Next, one can solve the Laplace equation for each of the stress components. Unlike the usual case of continuum mechanics, we have a unique solution without imposing constitutive relations and the equilibrium equations are not satisfied. For the harmonic solution of the boundary value problem, the maximal stresses occur on the boundary and these stresses are the smallest that satisfy the traction boundary conditions. Eq. (6.2) and its proof indicate that the maximal value of $\nabla \cdot \sigma$ is the smallest possible. In light of the usual equilibrium equations $\nabla \cdot \sigma + b = 0$ of continuum mechanics (b being the body force field), we can interpret the field $-\nabla \cdot \sigma$ as additional body forces one has to supply for the equilibrium condition to hold. Thus, for equilibrium, the harmonic stress field is associated with an additional body force field whose maximum is the least (and is attained on the boundary). It is noted that the total of the traction field, $\int_{\partial \Omega} t$, was not required to vanish so it is not possible for equilibrium to hold. With the foregoing limitations in mind, the harmonic stress field solves the problem of optimal stress field for a given traction.

Next, we note that the generalized stress concentration factor may be described as the largest optimal stress concentration factor when we can vary the boundary traction fields while keeping their maximal value on the boundary to $\|t\|_{\infty,\partial\Omega} = 1$. Thus, in the analysis all the components of the traction fields are set to be 1 everywhere. Again, this precludes equilibrium as the total force on the body in each direction is equal to the area of the boundary. The mathematical analog of this limitation is that we obtain bounds on γ and not $\gamma/\mathscr{R}$. However, the bound on $\|\gamma\|$ gives a bound $\|\gamma/\mathscr{R}\|$ because $\|\gamma/\mathscr{R}\| \leqslant \|\gamma\|$ (Eq. (2.22)).

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